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Résumé

Le Natyashastra, un traité sanscrit sur la musique, la danse et le théatre, contient une théorie des échelles musicales qui sert de base à la classification de formes mélodiques. De nombreux auteurs indiens et occidentaux ont proposé des interprétations de cette théorie, et plus récemment des travaux ont été menés pour valider ces interprétations à partir de mesures acoustiques. Cet article énonce les raisons pour lesquelles une telle validation n'est pas possible, et propose une interprétation générale de la théorie s'appuyant sur la résolution de systèmes d'équations linéaires. Les systèmes déduits des prémisses ont un degré d'indétermination. Les ensembles solutions correspondent à diverses hypothèses permettant de lever l'indétermination, tirées de ~~lois~~ relations acoustiques définissant diverses conceptions musicales: tempérament, intonation juste, etc.

Mots clés:
Ethnomusicologie - acoustique musicale - gammes - tempérament - accordage d'instruments

Keywords:
Ethnomusicology - musical acoustics - scales - temperament - instrument tuning

A Mathematical Discussion of the Ancient Theory of Scales according to Natyashastra

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GRTC/ 272 /mars 1988
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A MATHEMATICAL DISCUSSION OF THE ANCIENT THEORY OF SCALES ACCORDING TO *NATYASHASTRA*

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February 1984
revised March 1988 and December 2020

INTRODUCTION

The first six chapters of Bharata's *Natyashastra*, a Sanskrit treatise on music, dance and drama dating back to a period between 400 BC and 200 AD, contain the premises of a scale theory which caught the attention of scholars in India and in Western countries. Early interpretations by Western theorists followed the 'discovery' of this text by Sir Williams Jones (1784), and indeed Helmholtz's theory of 'natural' consonance gave way to many comparative attempts. Work by Indian scholars should not be undervalued, however, and among publications in English language the most complete study of the ancient system remains Bose's *Melodic Types of Hindustan* (1960).

In the early 20th century, Clements and Deval made experiments on a monochord with the great vocalist Abdul Karim Khan, thus opening the way to possible justifications of the ancient system at the light of actual music performance. Their interpretation of the system of twenty-two *shruti*-s as a system of pentadic harmony gained ground quickly because it was 'voted' to be the correct solution to the problem by the All India Music Conference held in 1929 in Madras [cited from Arnold 1980]. It never gained the support of Pt Bhatkhande and his followers, however [Bhatkhande H.S.P. IV:31], and even Abdul Karim Khan [Karim Khan 1968:75] later broke with Clements and Deval.

More recently, A. Daniélou (1943), B.C. Deva (1967), N.A. Jhairazbhoy (19(3), M. Levy (1982) and others designed their own experimental set-ups, coming to contradictory conclusions drawn from the analysis of small samples of music performance. It was only after 1981 that systematic experiments were conducted in India by the ISTAR team (J. Arnold, B. Bel, J. Bor and W. van der Meer) with an electronic programmable harmonium (the *Shruti Harmonium*) and a kind of 'microscope' for melodic music, the *Melodic Movement Analyser* (MMA) [Arnold & Bel 1983, Bel and Bor I985], feeding data to a computer to process hours of music samples selected from famous recordings.

It has become clear, after several years of experimental work, that although intonation in present day Indian music performance is far from being a random process, it would be hazardous to assess an interpretation of ancient scale theory with the aid of contemporary musical data. There at least three reasons for this:

1) There are infinitely equally valid interpretations of the ancient theory, as this paper will point out.
2) The concept of raga, i.e. the basic principle of Indian classical music, appeared first in the literature circa 900 AD in Matanga's *Brihaddeshi*, and it underwent a gradual development until 13th century, when Sharangadeva enlisted 264 ragas in his *Sangitratnakara*.
3) Drones were not in use at the time of *Natyashastra*; the influence of the drone on intonation is considerable, if not predominant, in contemporary music performance.

The latter argument reveals much of the split between musical theory and practice. Although many musicians with a strong traditional background still consider the inner consonance of scales an essential phenomenon, they rarely check mutual consonance when confronted with tuning problems, e.g. tuning resonating strings on the sitar or *sarangi*. Tuning experiments with the *Shruti Harmonium* have shown that expert musicians even feel confused if the experimenter points out dissonance where they expected perfect consonance, not being fully aware of distortions caused by the attractive effect of the drone.

Consequently, the aim of this work is not to provide normative results that could be applied to a "standardisation of *shruti*-s", as many scientific (?) studies have claimed. The ancient theory is useful in the way that it provides insight into early melodic classification (the *jati* system) which later was adapted to the raga system, but it should be understood as a topological description of scale relations. To support this view, this paper investigates the various sets of tuning systems that can be derived from the axiology of *Natyashastra*. These systems share interesting properties with respect to inner consonance and consistency in plagal shifts, thereby answering problems that led to the adoption of equal temperament in the West.

Since it has become possible nowadays to tune any scale accurately with the aid of electronic devices, these tuning systems may also provide a new framework for hobbyist musicians who wish to experiment with modal music.

INTERVALS

According to *Natyashastra*, musical scales are based on a division of the octave into 22 microintervals or '*shruti*-s'. Nowhere in the text are the sizes of these *shruti*-s defined with the aid of integer ratios or lengths of vibrating strings, etc.

As a consequence, two classes of interpretations are prevailing. One is that these microintervals are equal in size, so that the system of 22 *shruti*-s is nothing else that an equal temperament in 22 microtones [Te Nijenhuis 1974]. The opposite theory states that Bharata's scales are based on tuning procedures according to the system of *just intonation*, i.e. based on perfect fifths, fourths, and harmonic major thirds [Arnold 1980].

Tuning experiments with the *Shruti Harmonium* pointed at a reality somewhere between these two interpretations. First we need to consider subjective intervals : it is known that fifths and octaves may be stretched so that theoretical frequency ratios are generally not respected by sensitive tuners. It is not easy, therefore, to tune scales consistently, with the result that the tuning procedure referred to by the ancient theory may be a combination of the two following methods:

1) Tune twenty-two strings in equal temperament. This can be achieved by correcting string tunings in a systematic way, listening to the same melodic pattern which is shifted from one starting point to the next. However, as Bharata expected 13 *shruti* and 9 *shruti* intervals to be consonant, which this temperament does not allow (although he may not have noticed it), he may have proceeded to adjustments resulting in scales in which all fifths sound perfect but major thirds are slightly off. This might even be a reason why he does not classify the major third (7 *shruti*-s) as a 'consonant' interval (*samvadi*).

2) Tune cycles of fifths and fourths in 'just-intonation'. The tag in the system is that there are two cycles of fifths in Bharata's fundamental scales (*grama*-s) separated with a major third. Therefore it is necessary to define the major third.

Both methods arrive at similar results with the aid of an electronic device, but they lead to different distributions of errors if performed by ear. In any case the actual value of the major third remains hypothetical. Indeed, with the background of a drone the harmonic *Gandhara* (ratio 5/4) would be preferred. But, do we really know anything about other notes of the theoretical scale, such as for example *Komal Nisad* (B flat), which, accurately tuned in Pythagorean series (ratio 16/9), sounds 'too high' ?

Therefore, it would be unscientific to study Bharata's theory with the belief that it is an extension of 'just intonation'. Instead, we intend to proceed from Bharata's statements and 'translate' them to algebraic relations. Then we will study all possible solution sets.

We imagine that we have perfect electronic tools to measure frequency ratios. We also imagine that we are able to determine perceived pitch as no drone is interfering. In order to deal with linear systems we will express intervals in cents, a logarithmic unit defined as follows:

Let f1 and f2 be the frequencies of two tones, f1t2. The size of the interval between these two tones is: C = 1200 (log(f1/f2) / log(2)), where 'log' is a logarithm.

Since the focus is on theoretical scales, intervals ought to be associated to integer ratios. Howver, we will express them in cents with an accuracy much better than a trained ear would discriminate. We do not feel that ratios such as "531441/524288" bear more meaning than "21.5 cents".

We will label the sizes of the 22 *shruti*-s (starting from the base-note Sa): a, b, c, d… u, v. These letters represent measurements in cents. The first relation we can write is:

a+b+c+d+e+f+g+h+i+j+k+l+m+n+o+p+q+r+s+t+u+v = size of the octave = 1200 cents

which we call the 'octave equation'. To determine the values of a, b, c, d, etc. we need exactly 22 independent equations.

THE TWO BASIC SCALES: *GRAMA*-S

Bharata names the notes of the basic scale (*grama*) as follows: *Sadja*, *Rsabha*, *Gandhara*, *Madhyama*, *Pancama*, *Dhaivata* and *Nisada*, commonly abbreviated Sa, Re, Ga, Ma, Pa, Dha, Ni, and notated S, R, $\underline{G}$, M, P, D, $\underline{N}$. The reason why we underline $\underline{G}$ and $\underline{N}$ is that we will later define higher positions for these two notes. In *sargam* notation (the Indian solfege), lower note positions are named *komal* (flat) and the corresponding symbols are underlined.

Although we anticipate that S, R, $\underline{G}$, M, P, D and $\underline{N}$ are practically Do, Re, Mi, Fa, Fa, Sol, La and Si (i.e. C, D, E, F, G, A, B) of the Western scale, we only do this for the convenience of readers, since the choice of symbols has no incidence on results.

Bharata defines two scales named "Sa-*grama*" and "Ma-*grama*", differing only with the position of *Pancama*. There is an interval of one *shruti* (which he calls "*pramana shruti*") between *Pancama* of the Ma-*grama* and *Pancama* of the Sa-*grama*, the latter being higher than the former (XXVIII.24). For the sake of clarity, we will label P1 and P2 these two positions. Hence,

P1P2 = 1 *shruti*.

In XXVIII.22, Bharata indicates that P2 is 13 *shruti*-s above S (SP2 = 13), and M is 9 *shruti*-s above S (SM = 9). He further qualifies these 13 *shruti* and 9 *shruti* intervals as *samvadi*, which means 'consonant'. The highest consonance to be observed in musical scales, after the octave, is the perfect fifth (and its reverse interval the perfect fourth). Frequency ratios of such intervals are close to 3/2 and 3/4, that is 701.9 and 498.1 cents respectively. (Note that 701.9 + 498.1 = 1200.)

If SP2 = 13 *shruti*-s and SM = 9 *shruti*-s, then MP2 = 4 *shruti*-s.

Bharata also enumerates other *samvadi* relations in the basic scale:

P1R = 13 *shruti*-s, RD = 13 *shruti*-s and $\underline{GN}$ = 13 *shruti*-s.

He also says that S and P1 do not form a consonant interval, which we expected since:

SP1 = SP2 - P1P2 = 13 - 1 = 12 *shruti*-s.

Simple arithmetics will establish that:

RP1 = 22 - 13 = 9 *shruti*-s
and SR = SP1 - RP1 = 12 - 9 = 3 *shruti*-s.

Since RD = 13 it can also be found that:

SD = SR + RD = 3 + 13 = 16 *shruti*-s.

Bharata further indicates (XXVIII.23) that 2 or 20-*shruti* intervals are dissonant, such as R$\underline{G}$ and D$\underline{N}$.

Hence,

S$\underline{G}$ = SR + R$\underline{G}$ = 3 + 2 = 5 *shruti*-s, and S$\underline{N}$ = SD + D$\underline{N}$ = 16 + 2 = 18 *shruti*-s.

The two basic scales (*grama*-s), are summarized in the following sequence of intervals:

S (3) R (2) $\underline{G}$ (4) M (3) PI (1) P2 (3) D (2) $\underline{N}$ (4) S

The corresponding intervals have the following sizes:

SR = a+b+c
R$\underline{G}$ = d+e
$\underline{G}\overline{M}$ = f+g+h+i
$\overline{M}$P1 = j+k+l
P1P2 = m
P2D = n+o+p
D$\underline{N}$ = q+r
$\underline{N}$S = s+t+u+v

At this stage, we have solved a 'topological' problem, i.e. locating the 22 microintervals between the different notes of the basic scales (Ma-*grama* and Sa-*grama*). Bharata himself gives an account of this solution in his statement: "Sa has 4 *shruti*-s, Re has 3, etc." (XXVIII.24), but this may be (and has been, see [Jones 1784)) misinterpreted as he does not indicate whether intervals should be counted *above* or *below* the tonal position. Wrong interpretations were attempts to identify the Sa-*grama* as the Zarlino scale, a C mode, whereas the two *grama*-s are D modes.

We cannot state that any of the pairs of microintervals (*shruti*-s) are equal in size, nor can we state (as many authors do) that sets of two, three, four or more *shruti*-s form equal intervals such as semitone, minor tone, major tone, etc…

EXPERIMENT OF THE TWO VINAS

The next source of information is the experiment of the 'two *vina*-s' described by Bharata in Chapter XXVIII.24. This is almost certainly a thought experiment as it is probably impossible to perform on mechanical instruments. Bharata suggests to take two *vina*-s (stringed instruments, some kind of harp) and tune them identically on Sa-*grama*.

How to tune Sa-grama?

If we consider everything we know about the two basic scales, we find that the tuning procedure cannot be established unambiguously: we are able to tune cycles of perfect fifths: P2, S, M, $\underline{N}$, $\underline{G}$, on the one hand, and we know that R and D form a perfect fifth on the other hand. But how do we locate R? We also know that RP1 is a perfect fifth, but about P1 (of the Ma-*grama*) we only know it is lower than P2 by a *pramana shruti*. Having tuned M and P2, Bharata perhaps determined the position of P1 by dividing the interval MP2 into four 'equal' parts. Otherwise he may have used a major third to obtain R or D. There are four intervals in the basic scales that might be interpreted as major thirds: RM (6 *shruti*-s), MD (7 *shruti*-s), DS (6 *shruti*-s) and $\underline{G}$P1 (7 *shruti*-s). If any of these intervals is recognized as a major third (the harmonic major third or any other one), Ma-*grama* and Sa-*grama* scales can be tuned accurately by ear. We will refer to each of these four options as TP1, TP2, TP3 and TP4 respectively. ('TP' stands for tuning procedure). The tuning procedure which consists of dividing MP2 into four quartertones will be referred to as TP5, and we will call TP6 a procedure by which P1 is taken 'slightly' lower than P2, i.e. with an undetermined size of the *pramana shruti*.

Once both *vina*-s are tuned on Sa-*grama*, Bharata instructs the experimenter to tune Pa of one of the *vina*-s (which we will refer to as 'moveable *vina*') on the Ma-*grama*. This amount to lowering Pa with a *pramana shruti*. How to proceed ? Either P1 has been obtained directly from P2 (procedures TP5 and TP6), or it should be taken a major third above $\underline{G}$.

Bharata then says: "The same (*vina*) (…) will be tuned in the *Sadja-grama*." (XXVIII.24) In other words, he requests the experimenter to restore the Sa-*grama* from the new *Pancama*. All tuning procedures defined above start with P2, and establish S, M and N as ascending fifths. Then tune R from M (TP1), or D from M (TP2), or D from S (TP3), or P1 from 6 (TP4), or P1 from P2 (TP5 and TP6). Tune the remaining notes, R or (and) D, using the relations RD = perfect fifth, and if necessary P1R = perfect fifth. It is obvious that, having used the same procedure from a different starting note, we tuned down all notes with the same interval, namely a *pramana shruti*.

It is practical to use a circular slide-rule to visualize potential note matches. As we do not have any information on the sizes of *shruti*-s we decide, for the moment, to draw them equal in size. But the reader should keep in mind that this model may also miss to show matches that occur. This can be understood if, for example, one of the two-*shruti* intervals is equal, or even smaller, to one of the three-*shruti* intervals. Therefore we will have to discuss both matches which seem to occur and matches which *do not* seem to occur.

The first model is given fig 1. Cut out the inner wheel and move it over the crown. We will use the following symbols:

N => M means that N of the moveable wheel is matching M of the fixed wheel.
P2 #> R means that P2 of the moveable wheel is *not* matching R of the fixed wheel.

Using this simple circular slide rule, we can visualize the first lowering of the experiment as follows: first position the moveable wheel on the crown so that both S's are superposed (the two *vina*-s are tuned identically). Then move the inner wheel anticlockwise until P2 of the inner circle matches P1 of the outer circle:

P2 => P1

Note that, at this stage, Bharata does not recognize any other note matching than P2 and P1. We assume there are none, otherwise he would have mentioned them as he does it systematically. We will write all facts we can derive from the first lowering, and the corresponding algebraic equations or inequations.

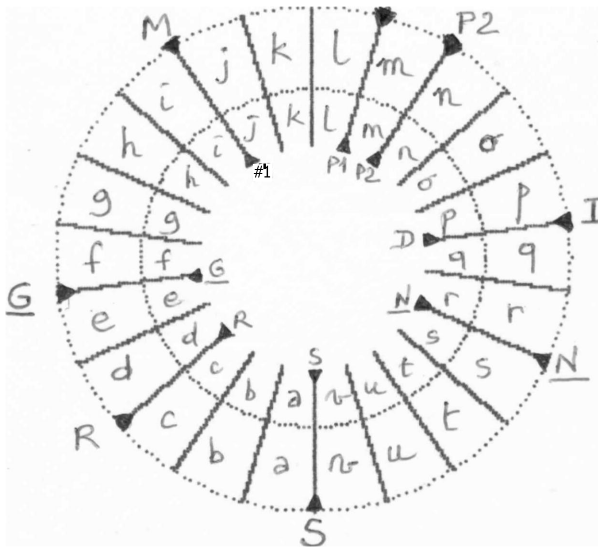


Fig. 1
(cut inner circle)

$\underline{G}$ #> R implies that $\underline{G}$ is still higher than R, not lower. This will be confirmed as $\underline{G}$ will match R after a second lowering. Hence the inequations:

s+t+u+v > m	S #> $\underline{N}$
a+b+c > m	R #> S
d+e > m	$\underline{G}$ #> R
f+g+h+i > m	M #> $\underline{G}$
n+o+p > m	D #> P2
q+r > m	$\underline{N}$ #> D

Second lowering:

In a similar way, one note of the variable *vina* is tuned down until it matches another note of the fixed one, and then all the other notes are tuned down so that Sa-*grama* is again established. This does not imply that the microinterval used in the second lowering is of the same size as the one used for the first one, as some authors have stated to conclude that Bharata "believed all *shruti*-s are equal in size" [See Jhairazbhoy].

The second lowering, as mentioned by Bharata, brings $\underline{G}$ on R and $\underline{N}$ on D. Therefore the total lowering has been q+r, or d+e. Hence,

q+r = d+e

Non-matching notes imply that:

f+g+h+i > d+e	M #> $\underline{G}$
a+b+c > d+e	R #> S
s+t+u+v > d+e	S #> $\underline{N}$
n+o+p > d+e	D #> P2
j+k+l+m > d+e	P2 #> M

Third lowering:

Now the moveable *vina* is tuned down so that:

R => S and D => P2

Therefore the total lowering has been (a+b+c) or (n+o+p). Hence:

n+o+p = a+b+c

s+t+u+v > a+b+c	S #> $\underline{N}$
f+g+h+i > a+b+c	M #> $\underline{G}$
j+k+l+m > a+b+c	P2 #> M

Fourth lowering:

The moveable *vina* is again tuned down until:

P2 => M, M => $\underline{G}$, and S => $\underline{N}$

The total lowering has been (j+k+l+m), (f+g+h+i) or (s+t+u+v). Hence:

j+k+l+m = f+g+h+i
s+t+u+v = f+g+h+i

Bharata gives another piece of information, saying "… the same (one) *shruti* being decreased *Pancama*…", which means that this lowering is the same as the first one, from which we may derive the following equations:

d = f = j = s = m

ALGEBRAIC INTERPRETATION

We can summarize all facts inferred from the two-*vina* experiment, eliminating redundant equations and inequations. The set of facts established so far is the following:

(S1) $d+e > m$
(S2) $a+b+c > d+e$
(S3) $f+g+h+i > a+b+c$
(S4) $j+k+l+m = f+g+h+i$
(S5) $s+t+u+v = f+g+h+i$
(S6) $n+o+p = a+b+c$
(S7) $q+r = d+e$

Practically, (S1), (S2) and (S3) indicate that:

- any two-*shruti* interval is larger than the *pramana shruti*;
- any three-*shruti* interval is larger than any two-*shruti* interval;
- any four-*shruti* interval is larger than any three-*shruti* interval.

In other words, the terminology ‘one-*shruti*’, ‘two-*shruti*’ etc., amounts to ordering intervals on increasing sizes.

Still, we are left with 22 unknown variables and only 4 equations. These 22 variables can be ‘packed’ into a new set of 8 unknown variables which represent the ‘macro-intervals’, i.e. the steps of the *grama*-s. Now we need only 4 auxiliary equations to determine the scales. These may be deducted from acoustic information. First we express that the sum of the variables, the octave, is equal to 1200 cents:

(S8) $(a+b+c)+(d+e)+(f+g+h+i)+(j+k+l)+m+(n+o+p)+(q+r)+(s+t+u+v) = 1200$

Then we interpret all *samvadi* relationships as perfect fifths (ratio 3/2 = 701.9 cents):

(S9) $(a+b+c)+(d+e)+(f+g+h+i)+(j+k+l)+m = 701.9$ (SP2)
(S10) $(j+k+l)+m+(n+o+p)+(q+r)+(s+t+u+v) = 701.9$ (MS)
(S11) $(d+e)+(f+g+h+i)+(j+k+l)+m+(n+o+p) = 701.9$ (RD)
(S12) $(f+g+h+i)+(j+k+l)+m+(n+o+p)+(q+r) = 701.9$ ($\underline{\text{GN}}$)
(S13) $m+(n+o+p)+(q+r)+(s+t+u+v)+(a+b+c) = 701.9$ (P1R)

S1O, S11 and S12 can be derived from S9. These equations may therefore be discarded. We still need one more equation. As indicated above, it is necessary to know the size of the major third or that of the *pramana shruti*. For each tuning procedure (except TP5 and TP6) we propose two ‘ideal’ values for the major third: the harmonic major third (386.3 cents) and the Pythagorean major third (407.8 cents).

We arrive at the following options:

TP1H $(d+e)+(f+g+h+i) = 386.3$
TP1P $(d+e)+(f+g+h+i) = 407.8$
TP2H $(j+k+l)+m+(n+o+p) = 386.3$
TP2P $(j+k+l)+m+(n+o+p) = 407.8$
TP3H $(q+r)+(s+t+u+v) = 386.3$
TP3P $(q+r)+(s+t+u+v) = 407.8$
TP4H $(f+g+h+i)+(j+k+l) = 386.3$
TP4P $(f+g+h+i)+(j+k+l) = 407.8$

TP5 m = 1/4 (j+k+l+m) <=> (j+k+l)-3m = 0
TP6 m = … (any value)

From the preceding equations it is easy to prove that procedures TP1H and TP1P are equivalent to TP3H and TP3P respectively. In the same way, TP2H and TP2P are equivalent to TP4H and TP4P respectively. We are left with four distinct tuning options.

The system of independent linear equations is the following:

(S4) (j+k+l+m) - (f+g+h+i) = 0
(S5) (s+t+u+v) - (f+g+h+i) = 0
(S6) (n+o+p) - (a+b+c) = 0
(S7) (q+r) - (d+e) = 0
(S8) (a+b+c)+(d+e)+(f+g+h+i)+(j+k+l)+m+(n+o+p)+(q+r)+(s+t+u+v) = 1200
(S9) (a+b+c)+(d+e)+(f+g+h+i)+(j+k+l)+m = 701.9
(S13) m+(n+o+p)+(q+r)+(s+t+u+v)+(a+b+c) = 701.9
(S14) any of the major third equations, or the *pramana shruti* equation

The corresponding linear system will be presented in tables of coefficients, each column representing the coefficients of one of the unknown variables, and each line representing one equation. Below are the tables corresponding to different tuning options. On the bottom line of each table are the values of the unknown variables satisfying all equations. Each set of solutions is unique.

TP1H: tuning procedure #1, Harmonic third:

Equ	(a+b+c)	(d+e)	(f+g+h+i)	(j+k+l)	m	(n+o+p)	(q+r)	(s+t+u+v)		
S4	0	0	-1	1	1	0	0	0	=	0
S5	0	0	-1	0	0	0	0	1	=	0
S6	-1	0	0	0	0	1	0	0	=	0
S7	0	1	0	0	0	0	-1	0	=	0
S8	1	1	1	1	1	1	1	1	=	1200
S9	1	1	1	1	1	0	0	0	=	701.9
S13	1	0	0	0	1	1	1	1	=	701.9
S14	0	1	1	0	0	0	0	0	=	386.3
	111.75	182.4	203.9	111.75	92.15	111.75	182.4	203.9	<— Unique solution	

TP1P: tuning procedure #1, Pythagorean third:

Equ	(a+b+c)	(d+e)	(f+g+h+i)	(j+k+l)	m	(n+o+p)	(q+r)	(s+t+u+v)		
S4	0	0	-1	1	1	0	0	0	=	0
S5	0	0	-1	0	0	0	0	1	=	0
S6	-1	0	0	0	0	1	0	0	=	0
S7	0	1	0	0	0	0	-1	0	=	0
S8	1	1	1	1	1	1	1	1	=	1200
S9	1	1	1	1	1	0	0	0	=	701.9
S13	1	0	0	0	1	1	1	1	=	701.9
S14	0	1	1	0	0	0	0	0	=	407.8
	90.25	203.9	203.9	90.25	113.6	90.25	203.9	203.9	<– Unique solution	

TP2H: tuning procedure #2, Harmonic third:

Equ	(a+b+c)	(d+e)	(f+g+h+i)	(j+k+l)	m	(n+o+p)	(q+r)	(s+t+u+v)		
S4	0	0	-1	1	1	0	0	0	=	0
S5	0	0	-1	0	0	0	0	1	=	0
S6	-1	0	0	0	0	1	0	0	=	0
S7	0	1	0	0	0	0	-1	0	=	0
S8	1	1	1	1	1	1	1	1	=	1200
S9	1	1	1	1	1	0	0	0	=	701.9
S13	1	0	0	0	1	1	1	1	=	701.9
S14	0	0	0	1	1	1	0	0	=	386.3
	182.4	111.75	203.9	182.4	21.5	182.4	111.75	203.9	<– Unique solution	

TP2P: tuning procedure #2, Pythagorean third:

Equ	(a+b+c)	(d+e)	(f+g+h+i)	(j+k+l)	m	(n+o+p)	(q+r)	(s+t+u+v)		
S4	0	0	-1	1	1	0	0	0	=	0
S5	0	0	-1	0	0	0	0	1	=	0
S6	-1	0	0	0	0	1	0	0	=	0
S7	0	1	0	0	0	0	-1	0	=	0
S8	1	1	1	1	1	1	1	1	=	1200
S9	1	1	1	1	1	0	0	0	=	701.9
S13	1	0	0	0	1	1	1	1	=	701.9
S14	0	0	0	1	1	1	0	0	=	407.8
	203.9	90.25	203.9	203.9	0	203.9	90.25	203.9	<— Unique solution	

TP5: tuning procedure #5, quartertone:

Equ	(a+b+c)	(d+e)	(f+g+h+i)	(j+k+l)	m	(n+o+p)	(q+r)	(s+t+u+v)		
S4	0	0	-1	1	1	0	0	0	=	0
S5	0	0	-1	0	0	0	0	1	=	0
S6	-1	0	0	0	0	1	0	0	=	0
S7	0	1	0	0	0	0	-1	0	=	0
S8	1	1	1	1	1	1	1	1	=	1200
S9	1	1	1	1	1	0	0	0	=	701.9
S13	1	0	0	0	1	1	1	1	=	701.9
S14	0	0	0	1	-3	0	0	0	=	0
	152.8	141.4	203.8	152.8	51	152.8	141.4	203.8	<— Unique solution	

Examining these tables we can immediately reject procedure TP1 as it produces two-*shruti* intervals (e.g. d+e) which are larger than three-*shruti* intervals (e.g. a+b+c), in contradiction with the premises. TP2P is also incorrect since it yields a nil value for the *pramana shruti* (m).

A more general resolution, referred as TP6, would lead to the following values:

a+b+c = M - m

d+e = L + m

f+g+h+i = M

j+k+l = M - m

n+o+p = M - m

q+r = L + m

s+t+u+v = M

where L = 90.25 (lima) and M = 203.9 (major whole tone).

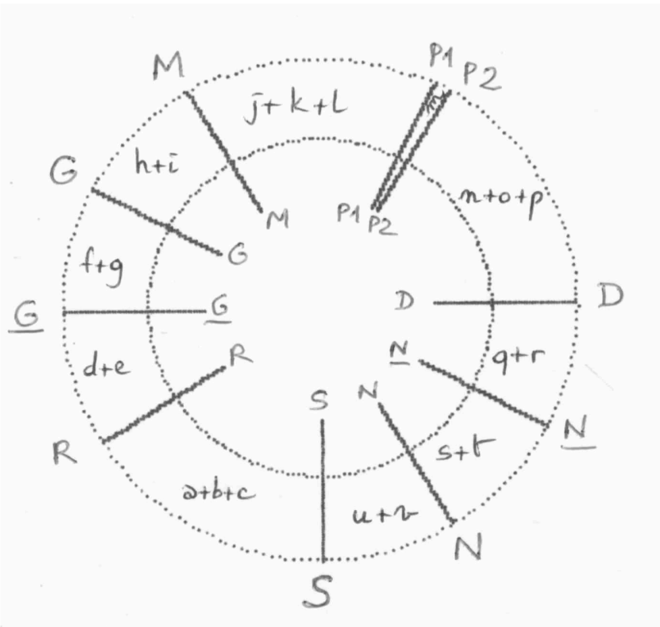
The range of variation for (m) is determined as follows: (m) is positive and (m) should be small enough to preserve the order of sizes (1 < 2 < 3). It is easy to notice that, if (m) grows, then 2-*shruti* intervals will increase and 3-*shruti* intervals will decrease. Hence, a maximum value for (m) is when:

L + m = M - m

==> m = 1/2 (M - L) = 0.5 * (203.9 - 90.25) = 56.8 cents.

We arrive at the conclusion that the correct tuning procedures are TP2 (take MD as a major third), TP4 (take GP1 as a major third), or TP5/TP6 in which the *pramana shruti* may take any value from 0 to 56.8 cents.

Indeed, one should pay attention to the particular case of a *pramana shruti* equal to 21.5 cents, as it is equivalent to using harmonic major thirds (ratio 5/4, or 386.3 cents) in TP2 or TP4. A model of the corresponding scale is shown fig 2:



Since the solution of TP2 is a particular case of that of TP6, we will use the solution set of TP6 as a general formula for possible tunings.

MURCCHANA-S

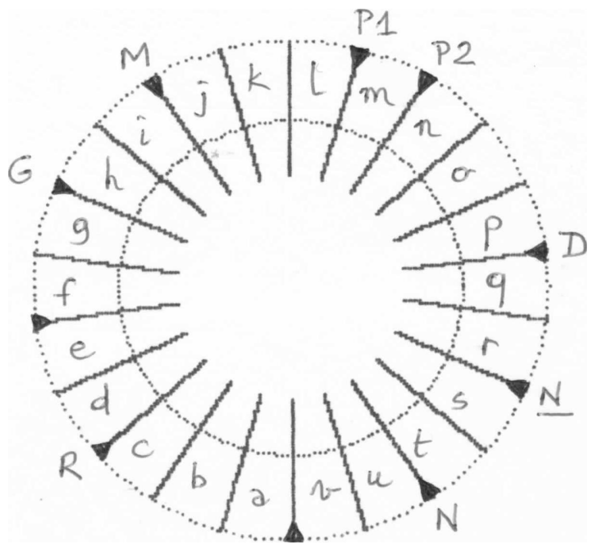
The experiment with the two *vina*-s is the practical demonstration of a 'plagal shift', namely *murcchana*: after the third lowering, Re of the moveable *vina* has reached Sa of the fixed one. Thus we are comparing the original scale with a new one obtained by shifting the tonic from Sa to Re. This is called the 'Re-*murcchana*' of the *grama*. Using the model, we can rotate the inner wheel until its Re reaches Sa of the outer circle. As seen earlier, *Dhaivata* on the inner wheel will match *Pancama* on the fixed crown. A fact which was not mentioned by Bharata, because he was dealing with the Sa-*grama*, P1 of Ma-*grama* will also match *Madhyama*.

We can try other *murcchana*-s as well, using G, M, etc. as starting notes, and notice that there occur as many matches with this model as with the 22-tempered-*shruti* scale (fig 1). This means that the system described by Bharata shows a maximum inner consistency without breaking laws of consonance.

ALTERED NOTES

In order to complete his system of scales, Bharata had to introduce two new notes in the basic *grama*-s: *antara Gandhara* and *kakali Nishada*. The new Ga, which we will notate G, is defined as G raised by 2 *shruti*-s. Similarly, kakali Ni, notated N, is N raised by two *shruti*-s. For the time being, we will mark these two notes on the simplified circular slide-rule (see fig 3).

In order to position N and G correctly we must investigate the behavior of the new scale in all *murcchana*-s (including those starting with G and N). Let us determine all equations corresponding to a perfect consistency of the scale (i.e, the same note matches as found on an equal-tempered circular model must be verified). Below is the set of equations derived from all *murcchana*-s:



Calculation covering all *murcchana*-s:

1	R	(a+b+c)+(d+e)+(f+g)+(h+i) = (d+e)+(f+g)+(h+i)+(j+k+l)	SM = RP1
2	R	(n+o+p)+(q+r)+(s+t)+(u+v) = (q+r)+(s+t)+(u+v)+(a+b+c)	DR = P2S
3	G	(a+b+c)+(d+e)+(f+g) = (f+g)+(h+i)+(j+k+l)	SG = GP1
4	G	(n+o+p)+(q+r)+(s+t)+(u+v) = (s+t)+(u+v)+(a+b+c)+(d+e)	NG = P2S
5	G	(u+v) = (d+e)	NS = RG
6	G	(a+b+c)+(d+e) = (h+i)+j+k+l)	SG = GP1
7	G	(a+b+c)+(d+e)+(f+g)+(h+i) = (h+i)+(j+k+l)+m+(n+o+p)	SM = GD
8	G	(n+o+p)+(q+r)+(s+t)+(u+v) = (u+v)+(a+b+c)+(d+e)+(f+g)	P2S = NG
9	G	(s+t)+(u+v) = (d+e)+(f+g)	NS = RG
10	G	(u+v) = (f+g)	NS = GG
11	M	(a+b+c) = (j+k+l)	SR = MP1
12	M	(a+b+c)+(d+e)+(f+g) = (j+k+l)+m+(n+o+p)	SG = MD
13	M	(a+b+c)+(d+e)+(f+g)+(h+i) = (j+k+1)+m+(n+o+p)+(q+r)	SM = MN
14	M	(n+o+p)+(q+r)+(s+t)+(u+v) = (a+b+c)+(d+e)+(f+g)+(h+i)	SM = P2S
15	M	(q+r)+(jt)+(u+v) = (d+e)+(f+g)+(h+i)	DS = RM
16	M	(s+t)+(u+v) = (f+g)+(h+i)	NS = GM
17	M	(u+v) = (h+i)	NS = GM
18	P1	(n+o+p)+(q+r)+(s+t)+(u+v) = (d+e)+(f+g)+(h+i)+(j+k+l)	P2S = RP1
19	P2	(a+b+c) = (n+o+p)	SR = P2D
20	P2	(a+b+c)+(d+e) = (n+o+p)+(q+r)	SG = P2N
21	P2	(a+b+c)+(d+e)+(f+g) = (n+o+p)+(q+r)+(s+t)	SG = P2N
22	P2	(a+b+c)+(d+e)+(f+g)+(h+i) = (n+o+p)+(q+r)+(s+t)+(u+v)	SM = P2S
23	P2	m+(n+o+p)+(q+r)+(s+t)+(u+v) = (d+e)+(f+g)+(h+i)+(j+k+l)+m	P1S = RP2
24	P2	(q+r)+(s+t)+(u+v) = (h+i)+(j+k+l)+m	DS = GP2
25	P2	(s+t)+(u+v) = (j+k+l)+m	NS = MP2
26	D	(a+b+c)+(d+e)+(f+g)+(h+i) = (q+r)+(s+t)+(u+v)+(a+b+c)	SM = DR
27	D	(n+o+p)+(q+r)+(s+t)+(u+v) = (h+i)+(j+k+l)+m+(n+o+p)	P2S = GD
28	D	(s+t)+(u+v) = m+(n+o+p)	NS = P1D
29	N	(a+b+c)+(d+e)+(f+g) = (s+t)+(u+v)+(a+b+c)	SG = NR
30	N	(a+b+c)+(d+e)+(f+g)+(h+i) = (s+t)+(u+v)+(a+b+c)+(d+e)	SM = NG
31	N	(n+o+p)+(q+r)+(s+t)+(u+v) = (j+k+l)+m+(n+o+p)+(q+r)	P2S = MN
32	N	(q+r)+(s+t)+(u+v) = m+(n+o+p)+(q+r)	DS = P1N
33	N	(u+v) = (q+r)	NS = DN
34	N	(a+b+c)+(d+e) = (u+v)+(a+b+c)	SG = NR
35	N	(a+b+c)+(d+e)+(f+g) = (u+v)+(a+b+c)+(d+e)	SG = N
36	N	(a+b+c)+(d+e)+(f+g)+(h+i) = (u+v)+(a+b+c)+(d+e)+(f+g)	SM = NG
37	N	(s+t)+(u+v) = (q+r)+(s+t)	DN = NS
38	N	(u+v) = (s+t)	NS = NN

This system can be reduced to the following set of independent equations:

1	(a+b+c) = (j+k+l)
2	(n+o+p) = (a+b+c)
5	(u+v) = (d+e)
10	(u+v) = (f+g)
17	(u+v) = (h+i)
33	(u+v) = (q+r)
38	(u+v) = (s+t)
7	(d+e)+(f+g) = m+(n+o+p)

Supplementing this system with *grama* equations S4, S5, S6, S7, S8, S9, S13 and S14, we arrive at eleven independent equations:

1, 2, 5, 10, 17, 33, 38, 7, S8, S9, S14

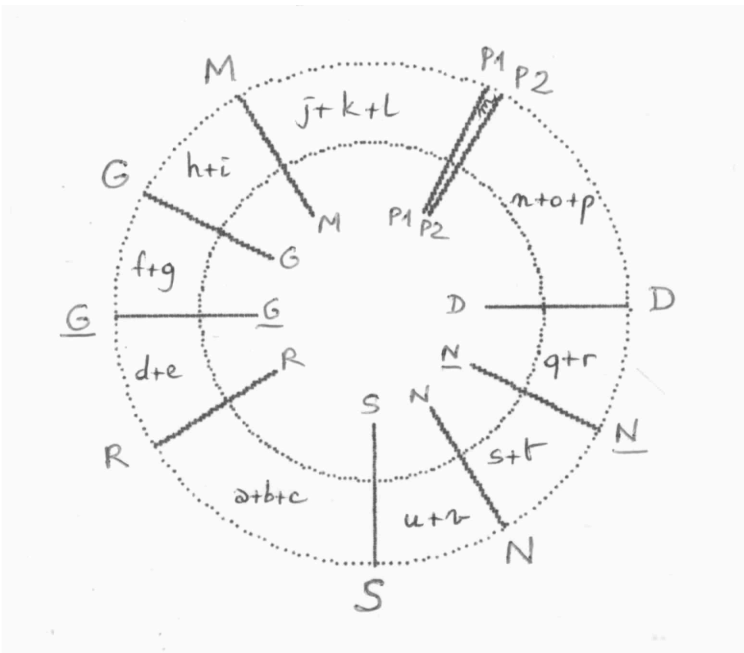
This system has only 10 unknown variables, and therefore no solution. The problem is to determine a meaningful compromise allowing to eliminate one of the above equations.

TEMPERED SYSTEMS

A first approach consists of ignoring S14, S9 or S8, thereby creating a kind of temperament on major thirds, fifths or octaves. Suppressing S14 yields the following system:

Equ	(a+b+c)	(d+e)	(f+g)	(h+i)	(j+k+l)	m	(n+o+p)	(q+r)	(sit)	(u+v)		
1	1	0	0	0	-1	0	0	0	0	0	=	0
2	-1	0	0	0	0	0	1	0	0	0	=	0
5	0	-1	0	0	0	0	0	0	0	1	=	0
10	0	0	-1	0	0	0	0	0	0	1	=	0
17	0	0	-1	-1	0	0	0	0	0	1	=	0
33	0	0	0	0	0	0	0	-1	0	1	=	0
38	0	0	0	0	0	0	0	0	-1	1	=	0
7	0	1	1	0	0	-1	-1	0	0	0	=	0
S8	1	- 1	1	1	1	1	1	1	1	1	=	1200
S9	1	1	1	1	1	1	0	0	0	0	=	701.9
	192	102	102	102	192	12	192	102	102	102	<— Unique solution	

The 7-*shruti* interval of this system has a constant value of 192 + 102 + 102 = 396 cents, i.e 10 cents above the harmonic major third and 12 cents less than the Pythagorean major third. The corresponding scale is labelled ‘system 1’ and represented figure 4:



Let us now determine other tempered tunings with a variable *pramana shruti*. At this point, we may reduce the system to the following independent equations:

a+b+c = j+k+l = n+o+p = L
d+e = f+g = h+i = q+r = s+t = u+v = D
m = C
(7) 2D - L = C
(S8) 6D + 3L = 1200 - C
(S9) 3D + 2L = 701.9 - C
(S14) 0 < C < 56.8

If we suppress S9, we get the solution:

D = 100 + 1/6 C
L = 200 - 2/3 C

Below is a table of intervals obtained with this system, considering typical values of C:

comma	semitone	limma	major third SG	major third GP2	fifth 8P2	octave
C	D	L	2D + L	2D + L + C	3D + 2L + C	6D + 3L + C
0	100	200	400	400	700	1200
21.5	103.6	185.7	392.8	414.3	703.6	1200
56.8	109.5	162.1	381	437.9	709.5	1200

Evidently, the first line of this table points to the equal-tempered system, but all values of C yield equally valid alternate temperaments.

If S8 is suppressed, we come to:

D = 1/7 (701.9 + C)
L = 2/7 (701.9) - 5/7 C

The corresponding intervals are calculated below:

comma	semitone	limma	major third SG	major third GP2	fifth SP2	octave
C	D	L	2D + L	2D + L + C	3D+ 2L + C	6D + 3L + C
0	100.3	200.6	401	401	701.9	1203.3
21.5	103.3	185.2	391.9	413.4	701.9	1197.1
56.8	108.4	160	376.8	433.6	701.9	1187

In general, octaves need to be tuned slightly larger than 1200 cents. Therefore, it would not be realistic to suggest tunings with tempered octaves if the comma is larger than 21.5

JUST SYSTEMS

If we maintain constraints on the values of the octave, the perfect fifth and the comma or *pramana shruti* (or equivalently on the major third SG) we must suppress one equation expressing inner consonance (S4, S5, S6 or S7) or some of the equations derived from *murcchana*-s (1, 2, 5, 10, 17, 33, 38 or 7). It is logical to maintain a maximum internal consonance by taking the best consistent *grama* previously defined, which led to the following result:

f+g+h+i = s+t+u+v = M
a+b+c = j+k+l = n+o+p = M - C
d+e = q+r = L + C
m = C

with L = 90.25 (limma) and M = 203.9 (major wholetone)

Additional independent equations drawn from *murcchana*-s are:

(5) u+v = L + C
(10) f+g = u+v
(17) h+i = u+v
(38) s+t = u+v
(7) f+g = M - L - C

This system has a unique solution (system 1) if C = 12. (This may be verified with equations 5, 10 and 7 that lead to L + C = M - L - C.) In general, either (5) or (7) must be deleted to provide a solution. The resolution of the system may be interpreted as follows: split major wholetones G̲M and N̲S in two intervals: L + C and M - L - C defining the positions of G and N. There is a unique solution maintaining perfect fifths GN and DG:

f+g = s+t = M - L - C
h+i = u+v = L + C

To summarize, just systems are determined as follows:

a+b+c = j+k+l = n+o+p = M - C
d+e = h+i = q+r = u+v = L + C
f+g = s+t = M - L - C
m = C

with L = 90.25 cents (limma), M = 203.9 cents (major wholetone)
and 0 < C < 56.8 (*pramana shruti* or comma)

Intervals are calculated below for typical values of C:

Comma C	M-C	L+C	M-L-C	major third SG 2 M - C (7 *shruti*-s)	major third G̲P2 2 M (8 *shruti*-s)	fifth	octave
0	203.9	90.3	113.7	407.8	407.8	701.9	1200
21.5	182.4	111.8	92.1	386.3	407.8	701.9	1200
56.8	147	147	56.8	351	407.8	701.9	1200

These just tunings are based on two cycles of perfect fifths and fourths separated by an interval depending on the value of the comma (generally a 7-*shruti* major third). Experiments with the Melodic Movement Analyser suggested that these tunings may match raga performance with a comma lower than 25 cents.

The just-intonation system using a harmonic major third (ratio 5/4) is achieved when C = 21.5 cents. This system will be labeled 'system 2' in the following discussion, and the corresponding scale is shown fig 5.

FURTHER DIVISIONS OF INTERVALS

This discussion deals with alternate representations of microtones and comparisons with other theoretical systems. Scale theories generally ignore temperament of octave and fifth as well as the idea of a variable comma. Therefore the two systems above are sufficient for comparison. Readers may use the two circular models (fig 4 and 5).

So far we have done no assumption on the sizes of *shruti*-s individually labelled: a, b, c…. As we were lacking information to establish 22 independent equations we have been induced to consider 'macro-intervals' (the steps of the *grama*-s), notated (a+b+c), (d+e), etc., as alternate variables.

Before we determine divisions of all steps in the *grama*-s, we should discuss whether or not such an operation is needed at all. Indeed, we could stay with the idea that numbers 1, 2 and 3 in '1-*shruti*', '2-*shruti*' and '3-*shruti*' are ordinals rather than cardinals. From this point of view, *grama*-s can be determined leaving aside the concept of *shruti* as a predefined interval. Musicians can tune and play scales, but generally they do not claim being able to sing or play the 22 *shruti*-s in order… *Shruti*-s are the 'subatomic particles' shaping tonal intervals.

Until now we have been using four basic intervals: a one-*shruti* interval, two two-*shruti* intervals, and a three-*shruti* interval. This system is far from perfect since it assigns two different sizes to two-*shruti* intervals. It is therefore more interesting to look for a smaller number of unambiguous micro-intervals describing scales.

Divisions using a single basic micro-interval

We can use a single microinterval to define scales, for example the cent. Can we use a larger unit? The greatest common divider of the (integer) values of the four intervals is:

System 1: 192, 102, 102, 12 -> 6
System 2: 182, 112, 92, 22 -> 1

With system 1, a one half comma unit would be ideal for measuring divisions of the octave. In system 2 no unit can be defined with an accuracy of 1 cent. Yet it can be noticed that:

182.3 / 21.5 = 8.48
111.8 / 21.5 = 5.2
92.1 / 21.5 = 4.28

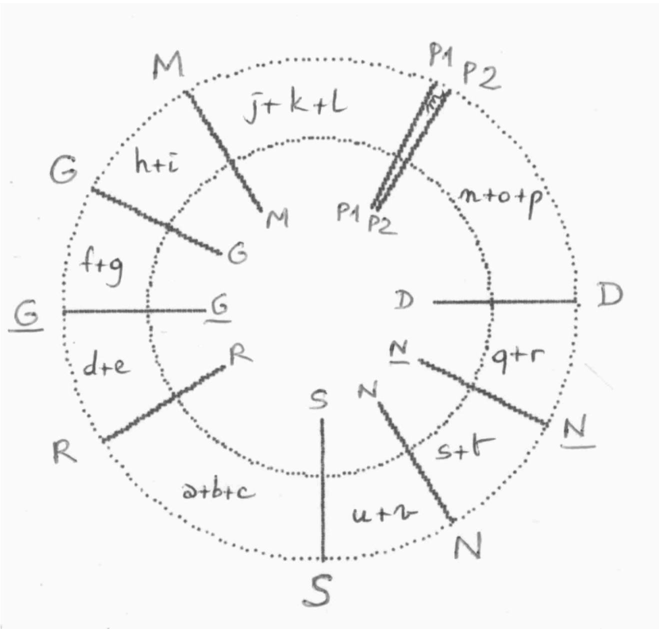


Figure 5

It is therefore possible to describe the 3-*shruti* and the two 2-*shruti* intervals of system 2 as respectively 8-comma, 5-comma and 4-comma intervals as done by Western theoreticians describing Zarlino's scale. More precisely, $8^{1/2}$, 5 and 4 should be used.

Divisions using two basic micro-intervals

Let us calculate differences between intervals in both systems:

System 1:

192 - 102 = 90
102 - 12 = 90

System 2:

182 - 112 = 70 (semitone)
182 - 92 = 90 (limma)
112 - 92 = 20
112 - 22 = 90 (limma)
90 - 70 = 20

We conclude that a simple system of two micro-intervals (12 and 90 cents) can be used for system 1, whereas the match is only approximate if we use 20 and 70-cent intervals for system 2.

Divisions using three basic micro-intervals

From the preceding discussion it is evident that we can use sets of 3 basic micro-intervals for describing both systems:

12, 90 and 102-cent intervals for system 1
22, 70 and 90-cent intervals for system 2,

namely, comma, minor semitone and limma.

These three intervals may be interpreted as the possible sizes of *shruti*-s, with the following interpretation in Western terminology:

'4-*shruti*': S + L + 2 C = major wholetone
'3-*shruti*': S + L + C = minor wholetone
'2-*shruti*': L + C
'2-*shruti*': L = limma or major semitone
'1-*shruti*': C = comma

CONCLUSION

Inferring algebraic relations from the statements on musical scales contained in the first chapters of *Natyashastra* led us to predict tuning systems entirely determined by the theory. System 1 (fig 4) is the one that best fits the stated requirements. Interestingly, this scale has never been envisaged in numerous speculations on *Natyashastra*. The corresponding tuning would be pleasing to the ear if performed on a harp without a drone.

Other tempered tunings may be envisaged, especially if the ear feels that octaves or (and) fifths need to be stretched. These tuning procedures may be used on digital synthesizers for the performance of modal music.

Just tuning, with two cycles of perfect fifths and an adjustable comma or 7-*shruti* major third, is closest to what Indian musicians claim to achieve: they care for inner consonance, but they generally do not decide on major thirds. Even though we can safely predict that SG would be close to 386 cents in ragas Bhupali or Kalyan, we would not predict the same with G̲P in raga Todi.

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